2018 22nd International Conference on System Theory, Control and Computing (ICSTCC)

Stabilization by Static Output Feedback: A Quantifier Elimination Approach

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***Abstract*—The controller design problem to stabilize a linear time-invariant state-space system by static state feedback has been solved decades ago. Combining the static state feedback with a state observer one obtains a dynamic output feedback control law. Contrary to that, the design problem of static output feedback stabilization is significantly more challenging. In this paper, we discuss the existence and the computation of stabilizing static output feedback control gains. Our approach allows additional specifications regarding the eigenvalues of the closed-loop system.**

***Index Terms*—stabilization, quantifier elimination, static output feedback**

I. Introduction

Many technical processes can be modeled appropriately by a system of linear time-invariant differential equations

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

with matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{r\times n}$. Therein u are the manipulated variables, y are the measured quantities, and x is the vector of states. A minimal requirement for a controller design is that the closed-loop system is asymptotically stable, i.e., the state vector x tends to the origin for an arbitrary initial value (corresponding to a disturbance) in the state space. This behavior can certainly be achieved by a *dynamic* output feedback if the system (1) is controllable and observable [1]. However, in some applications only a static output feedback (SOF)

$$u = Ky \tag{2}$$

with the gain matrix $K \in \mathbb{R}^{m\times r}$ can be realized. Then the question arises, whether we can force the closed-loop system

$$\dot{x} = (A + BKC)x. \tag{3}$$

to an arbitrary dynamic behavior. This problem was recently examined in [2] by a quantifier elimination approach. From a practical point of view, the problem formulation is on the one hand still too restrictive since one is usually only interested in the stabilization of the closed-loop system not in assigning arbitrary dynamics. On the other hand, in applications, additional constraints on the closed-loop eigenvalues will be imposed to achieve robust stability or quicker disturbance rejection, for instance. Both problems will be addressed in this paper.

We call system (1) *stabilizable by static output feedback* (2) if by an appropriate choice of K all eigenvalues of the closed-loop system matrix $A + BKC$ have negative real parts.

The remainder of the paper is structured as follows. In Section II recent results in the SOF stabilization are reviewed. Section III gives some basics on quantifier elimination (QE). The main Section IV first adopts the SOF stabilization problem formulation to be accessible to QE methods by an illustrative example and then provides results on stabilization, robust stabilization and interval stabilization by SOF.

II. Static Output Feedback Stabilization Problem

Stabilization of (1) via static output feedback (SOF) issues a still open problem in linear control theory [3], [4]. To give an overview, some results shall be mentioned in the following. Using a Lyapunov approach the SOF problem can be stated as follows.

Theorem 1: System (1) is stabilizable via static output feedback (2) iff there exists a symmetric positive definite matrix P of appropriate dimension such that the *Lyapunov equation*

$$P(A + BKC) + (A + BKC)^T P + I = 0 \tag{4}$$

holds.

The positive definiteness of the matrix P is denoted by $P \succ 0$. From a numerical point of view it is often practical to replace the Lyapunov equation by an appropriate matrix inequality:

Theorem 2: System (1) is stabilizable via static output feedback (2) iff there exists a symmetric positive definite matrix P of an appropriate dimension such that the matrix inequality

$$P(A + BKC) + (A + BKC)^T P \prec 0 \tag{5}$$

holds.

An equivalent representation to (5) is

$$(A + BKC)W + W(A + BKC)^T \prec 0 \tag{6}$$

with $W = P^{-1}$. Obviously, for a fixed K or fixed P, inequalities (5) and (6) become *linear matrix inequalities (LMIs)*, for which various solvers exist. However, solving (5)

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or (6) for K and P respectively W simultaneously is a much more challenging task [4] due to the bilinearity.

As suggested in [4], [5] multiplying (5) on the left by $(C^{\perp})^T$ and on the right by $((C^T)^{\perp})^T$ and multiplying (6) on the left by $B^{\perp}$ and on the right, by $(B^{\perp})^T$ results in

$$(C^T)^{\perp}(PA + A^T P)((C^T)^{\perp})^T \prec 0, \tag{7a}$$

$$B^{\perp}(AW + WA^T)(B^{\perp})^T \prec 0, \tag{7b}$$

where $B^{\perp}$ and $(C^T)^{\perp}$ are orthocomplements to B and C^T respectively. Thus Theorem 2 is equivalent to the existence of matrices P and W with $W = P^{-1}$ such that the inequalities (7) hold [5]. Although the bilinearity has been resolved, solving the coupled LMIs (7) is a non-convex problem where standard LMI techniques cannot be used. However, several algorithms had been developed in order to solve (7), see [6]–[8], where all these algorithms provide sufficient conditions only as convergence cannot be guaranteed.

Finsler's Lemma [9] allows for writing (7) in variable P only, resulting in

$$PA + A^T P - \alpha C^T C \prec 0, \tag{8a}$$

$$PA + A^T P - \beta PBB^T P \prec 0, \tag{8b}$$

where $\alpha, \beta \in \mathbb{R}$. Only for $\beta < 0$, which we cannot assume in general, the quadratic inequality (8b) can be rewritten as a LMI using the Schur complement [10]. Hence, this approach does also not lead to an applicable formulation. A necessary and sufficient condition is given in [11]. Unfortunately, this condition is difficult to check as well. Due to this, an alternative condition is given which is easier to test but also only sufficient. In [12] it is suspected that the SOF stabilization problem is highly complex and not suitable to be formulated in terms of LMIs.

An alternate approach, resulting in a necessary and sufficient condition to assign desired eigenvalues was given in [13]. Although the condition is hard to check, it also provides a possibility for calculating feedback matrices with low gain. For other research relating to eigenvalue assignment see [14]–[16] and the references therein.

III. Quantifier Elimination

In control theory and control system design we often face problems containing quantified variables, such as: "There exists ($\exists$) a parameter ... such that for all ($\forall$) ...". Quite often, these conditions are difficult to solve due to the inherent degree of freedom. To illustrate what is meant we consider the scalar system $\dot{x} = -x$. The asymptotic stability of the system can be investigated following the ideas of Lyapunov's direct method. We are looking for a positive definite function V with a negative definite time-derivative $\dot{V}$. Considering $V(x) = px^2$ we can formulate the before propositions in a mathematical notation

$$\begin{aligned} \exists p \forall x : \; & [x = 0 \implies (V(p,x) = 0 \wedge \dot{V}(p,x) = 0)] \\ & \wedge [x \neq 0 \implies (V(p,x) > 0 \wedge \dot{V}(p,x) < 0)]. \end{aligned} \tag{9}$$

It is straightforward to show that there exist values for p which fulfill the given requirements. Generally, the pure existence statement is not sufficient, rather the concrete values of p are of interest. How values satisfying given conditions of the form (9) can be computed in an automatic manner, is the topic of this section.

Before explaining the used concepts and tools, let us briefly introduce some definitions and notations. Formula (9) is a so-called *prenex* formula. Such prenex formulas are described by

$$G(y, z) := (Q_1 y_1) \cdots (Q_l y_l)\, F(y, z) \tag{10}$$

with $Q_i \in \{\exists, \forall\}$. The formula $F(x, y)$ is called *quentifier-free* and is a boolean combination of polynomials with rational coefficients

$$\varphi(y_1 \ldots, y_l, z_1, \ldots, z_k)\, \tau\, 0,$$

with $\tau \in \{=, \neq, <, >, \leq, \geq\}$. These formulas φ are called *atomic* formulas. Due to the fact that the variables y are connected with a quantifier, they are called *quantified* variables and the variables z are called *free*, respectively. Applied to the short example (9) we have the quantified variables p, x and no free variable since we are interested in a `true/false` result. In the general context, we are interested in a quantifier-free equivalent $H(z)$ to prenex formula (10). Note that the resulting quantifier-free formula $H(z)$ only depends on the quantifier-free variables z. Now two questions arising: Does such a quantifier-free equivalent always exists? and If it exists, how can it be determined?

The first question is directly answered with the following theorem [17, pp. 69-70].

Theorem 3 (Quantifier Elimination over the Real Closed Field): For every prenex formula $G(y, z)$ there exists an equivalent quantifier-free formula $H(z)$.

Thus the sought $H(z)$ always exists and the first question can be answered with `yes`. The second question leads directly to the concept of *quantifier elimination (QE)*. In the 1940s, the polish-american mathematician Alfred Tarski proved that over the real field there exists always a quantifier-free formula which is equivalent to a prenex formula [18] and presented the first algorithm to compute this quantifier-free equivalent. However, the computational complexity cannot be bounded by any stack of exponentials and was thus not applicable to non-trivial problems.

During the last decades, several strategies to solve practical QE problems have been developed. Starting with the *cylindrical algebraic decomposition* (CAD) [19] in the 1970s, followed by *virtual substitution* [20]–[22] and *real root classification* [23]–[25]. These algorithms and all its modifications and further developments do still face inherent computational barriers. Nevertheless, smart problem formulations and efficient implementations allow the solution of practical QE problems. To open these algorithms to a broad community some software tools have been developed. The first tool to solve non-trivial problems was the open-source package QEPCAD (Quantifier Elimination by Partial Cylindrical Algebraic Decomposition) [26]. This had been refined to QEPCAD B [27], which is available to most of the repositories of common Linux distributions. Furthermore, algorithms based on virtual substitution are implemented in the Open-Source-Packages REDUCE and REDLOG [28], [29]. This approach often yields large expressions, that can be simplified by the tool SLFQ [30] using QEPCAD B. For the computer algebra system Maple there exists the RegularChains library [31], which contains very efficient implementations of CAD-based algorithms. Another tool available for Maple is the software package SyNRAC [32], which contains efficient implementations of CAD, virtual substitution and real root classification based algorithms.

To our knowledge, the first application of quantifier elimination in the control of continuous time systems deals with the stabilization of linear system [33]. Other publication on the stabilization with quantifier elimination using the Routh, Hurwitz or Liénard-Chipart test can be found in [34], [35]. Modern approaches rely on formulations in terms of Lyapunov equations to carry out the stabilization [36], [37]. The design of robust controllers is discussed in [38], [39]. The formal verification of stability of nonlinear systems using Lyapunov functions is treated in [40]. This approach has been extended to input-to-state stability in [41] and to the computation of invariant sets using Lyapunov techniques in [42].

IV. QE Approach to Static Output Feedback

A. Introduction Example

We will discuss various scenarios for controller design by static output feedback using quantifier elimination. The design procedures will be illustrated using the example taken from [33] with the matrices

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix}, & B &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{pmatrix}, & K &= \begin{pmatrix} k_1 & k_2 \end{pmatrix} \end{aligned} \tag{11}$$

of the closed-loop system (3). The associated characteristic polynomial is given by

$$\begin{aligned} \mathrm{CP}(s) &= \det(sI - (A + BKC)) \\ &= a_0 + a_1 s + a_2 s^2 + s^3 \\ &= k_2 + (k_2 - 5k_1 - 13)s + k_1 s^2 + s^3. \end{aligned} \tag{12}$$

Since the number of entries in the gain matrix K is strictly less than the dimension $n = 3$ of the state-space, arbitrary eigenvalue placement will not be possible. Our aim is the stabilization of the closed-loop system (3) through an appropriate choice of the gain matrix K. To this end we have to check whether the characteristic polynomial (12) is a Hurwitz polynomial. The Liénard-Chipart test

$$a_0 > 0 \wedge a_2 > 0 \wedge a_1 a_2 - a_0 > 0 \tag{13}$$

yields

$$k_1 > 0 \wedge k_2 > 0 \wedge k_1 k_2 - k_2 - 5k_1^2 - 13k_1 > 0. \tag{14}$$

In [33] the authors raised the questions whether the system (14) of nonlinear inequalities has a real solution and how the solution can be computed. Both questions can be investigated using QE. In [33], QE was carried out manually (i.e., by hand), whereas in [4], the tool QEPCAD was used.

The question regarding to the existence of a real solution of (14) can be formulated as the prenex formula

$$\exists k_1, k_2 : \quad \text{Cond. (14)}.$$

This formula has no free variables, i.e., all variables are quantified. Carrying out QE yields the equivalent quantifier-free expression `true`, which confirms the existence of a real solution.

In order to compute a specific solution, we omit the existence quantifier for one variable, compute the feasible interval, specify the variable accordingly and proceed with the next variable. As in [4], [33] we first omit the existence quantifier for k_1 resulting in the prenex formula

$$\exists k_2 : \quad \text{Cond. (14)}$$

with the free variable k_1. QE results in the quantifier-free expression $k_1 > 1$. A possible choice is $k_1 = 2$, from which together with (14) we obtain $k_2 > 46$. With $k_2 = 50$ this condition holds. Hence, the closed-loop system (3) is stable (see Table I for the eigenvalues).

B. Stabilizability

Stability and stabilizability of a linear state-space system can easily be checked algebraically using Lyapunov techniques. Based on Theorem 1, the stability of the closed-loop system (3) with a given gain matrix K can be formulated as

$$\exists P : \quad P \succ 0 \wedge \text{Eq. (4)}.$$

Similarly, the question regards the stabilizability of system (1) can be stated as

$$\exists P, K : \quad P \succ 0 \wedge \text{Eq. (4)}.$$

The matrix-valued Lyapunov equation (4) is equivalently stated as the conjunction of scalar equations. The definiteness of the matrix P can be verified using Sylvester's law of inertia. This approach leads to a large number of existence quantifiers associated with the entries of the matrix P. Details the evaluation of these conditions using quantifier elimination can be found in [36], [37].

In a similar manner as in [4], [33] we can verify stability and stabilizability for linear systems with characteristic polynomials of arbitrary order. Each real polynomial can be decomposed into linear factors. If the polynomial has a conjugate pair, this decompositions results in complex coefficients. In order to obtain a purely real decomposition, we have to include quadratic factors. Every even polynomial can be decomposed into a product of quadratic factors. For odd polynomials, we take a further linear factor into account. For the example system given by (11) with $n = 3$, every desired characteristic polynomial can be decomposed as

$$\begin{aligned} \mathrm{CP}^*(s) &= a_0^* + a_1^* s + a_2^* s^2 + s^3 \\ &= (s^2 + bs + c)(s + d) \end{aligned} \tag{15}$$

with real coefficients $b, c, d \in \mathbb{R}$. Such a decomposition into polynomials of degree 1 and 2 allows a simple stability test. Linear and quadratic polynomials are Hurwitz polynomials iff their coefficients have the same signs.[1] Hence, the characteristic polynomial (15) is Hurwitz iff

$$b > 0 \wedge c > 0 \wedge d > 0. \tag{16}$$

Therefore, the system described by (11) is stabilizable by static output feedback iff

$$\exists b, c, d, k_1, k_2 : \quad \text{Cond. (16)} \wedge \text{CP} \equiv \text{CP}^*, \tag{17}$$

where the polynomials are compared coefficient-wise, i.e.,

$$\text{CP} \equiv \text{CP}^* \quad \Longleftrightarrow \quad a_0 = a_0^* \wedge \ldots \wedge a_{n-1} = a_{n-1}^*.$$

To obtain conditions for gain coefficients k_1, k_2 that stabilize the closed-loop system we omit the associated existence quantifiers and obtain the prenex formula

$$\exists b, c, d : \quad \text{Cond. (16)} \wedge \text{CP} \equiv \text{CP}^*. \tag{18}$$

We used the computer algebra software REDUCE to carry out QE regarding to the quantified variables b, c, d resulting in 2.1 KiB ASCII source code for the equivalent quantifier-free expression. Simplification with SLFQ yields

$$k_1 - 1 > 0 \wedge k_1 k_2 - k_2 - 5k_1^2 - 13k_1 > 0 \tag{19}$$

after 91 QEPCAD calls. The result (19) is equivalent to condition (14). The controller gains can be computed as sketched in Section IV-A starting with k_1 as a free variable. We obtain the same results as above.

Remark 1: In order to compute a solution of (19) we could also start with k_2 as a free variable and k_1 as a quantified variable, respectively. The equivalent quantifier-free formula obtained from QE is

$$k_2 - 13 > 0 \wedge k_2^2 - 46k_2 + 169 > 0. \tag{20}$$

The quadratic polynomial has the largest real root $k_2 = 23 + 6\sqrt{10} \approx 41.97$. The choice $k_2 = 42$ fulfills condition (20) and results in $2.8 < k_1 < 3$. With $k_1 = 2.9$ we obtain a gain matrix with a significantly smaller Frobenius norm compared to [4], [33] and Section IV-A.

C. Robust Stabilizability

To achieve robustness we want to place the poles not only in the complex left half plane but also ensure a distance to the imaginary axis. To achieve this goal, we substitute $s \mapsto s - \alpha_0$ in the desired characteristic polynomial with a given stability margin $\alpha_0 < 0$:

$$\text{CP}^*(s) = \big((s-\alpha_0)^2 + b(s-\alpha_0) + c\big)\,((s-\alpha_0) + d)\,. \tag{21}$$

The real part of all roots of the polynomial (21) is strictly less than $\alpha_0 < 0$ iff condition (16) holds. The system is robustly stabilizable with the stability margin $\alpha_0 < 0$ iff condition (17) holds with the polynomial (21). To obtain conditions on the entries of the gain matrix we evaluate (18) with (21). For example, the margin $\alpha_0 = -1$ results in the condition

$$k_1 - 3 > 0 \wedge k_1 k_2 - 3k_2 - 7k_1^2 + 5k_1 + 18 > 0. \tag{22}$$

We select $k_1 = 4$ and proceed with k_2 as a free variable resulting in $k_2 > 74$. With $k_2 = 75$, the real part of all three eigenvalues is strictly less than -1.

To answer the question, how far the eigenvalues of the closed-loop system can be moved to the left by static output feedback we take $\alpha_0 < 0$ as a free variable and associate the gains k_1, k_2 with the existential quantifier. QE with REDUCE and SLFQ yields the quantifier-free expression

$$\alpha_0 < 0 \wedge \alpha_0^3 + 3\alpha_0^2 - 15\alpha_0 + 13 > 0. \tag{23}$$

The cubic polynomial has one real root $\alpha_0 = -12^{1/3} - 6 \cdot 12^{-1/3} - 1 \approx -5.91$. For $s_0 = -5.9$, QE reduces (25) to `true`, whereas for $\alpha_0 = -6$ we obtain `false`.

D. Real Stabilizability

A stable system can still have a complex conjugate pair of eigenvalues (with negative real parts). In the time domain, such an eigenvalue configuration corresponds to (although declining) oscillations. Hence, we investigate the question whether for a given system (1) can be stabilized with purely real eigenvalues. In this case, the characteristic polynomial has only real roots and can therefore be decomposed into real linear factors. For our example system with $n = 3$ we obtain the decomposition

$$\text{CP}^*(s) = (s - s_1)(s - s_2)(s - s_3) \tag{24}$$

with the roots $s_1, s_2, s_3 \in \mathbb{R}$. The decomposed polynomial (24) is a Hurwitz polynomial iff $s_1, s_2, s_3 < 0$. The stabilizability with purely real eigenvalues can be stated by the prenex formula

$$\exists s_1, s_2, s_3, k_1, k_2 : \; s_1 < 0 \wedge s_2 < 0 \wedge s_3 < 0 \wedge \text{CP} \equiv \text{CP}^*,$$

where QE yields the result `true`. If we use k_1 as a free variable while k_2 remains quantified, QE with REDUCE results in 16 KiB ASCII source code for the equivalent quantifier-free expression. A simplification with SLFQ generates the condition

$$k_1^3 - 9k_1^2 - 135k_1 - 351 \geq 0,$$

which implies $k_1 \geq 18^{2/3} + 3 \cdot 18^{1/3} + 3 \approx 17.73$. We set $k_1 = 18$ and take k_2 as a free variable. This results in

$$209.9127 \lessapprox k_2 \lessapprox 210.0691.$$

With $k_2 = 210$ we have a stabilizing controller gain resulting in real eigenvalues for the closed-loop system (3).

[1] The sign condition is necessary but not sufficient for the Hurwitz property of polynomials with degree $n \geq 3$.

E. Robust Real Stabilizability

Now, we want to design the static output feedback such that the matrix of the closed-loop (3) system has purely real eigenvalues in the left half plane with a certain distance to the imaginary axis. For a prescribed value $\alpha_0 < 0$ this problem can be formulated as

$$\exists s_1, s_2, s_3, k_1, k_2 : \\ s_1 \leq \alpha_0 \wedge s_2 \leq \alpha_0 \wedge s_3 \leq \alpha_0 \wedge \alpha_0 < 0 \wedge \mathrm{CP} \equiv \mathrm{CP}^* \tag{25}$$

with CP^* from (24). Similar as above we can calculate the regions for the gains k_1, k_2. E.g., the gain computed in Section IV-D fulfills (25) with $\alpha_0 = -5$.

Furthermore, we are able to compute the minimum value of $\alpha_0 < 0$, which corresponds to the largest distance to the imaginary axis. Taking α_0 as a free variable, QE yields the closure of the set given by (23). This means we have the same margin $\alpha_0 \approx -5.91$ as in Section IV-C.

F. Real Interval Stabilizability

For instance, if we want to place two eigenvalues in the interval $[-5, -1]$ and one in $[-10, -1]$, this problem can be described by the prenex formula

$$\exists s_1, s_2, s_3, k_1, k_2 : \; -5 \leq s_1 \leq -1 \wedge \\ -5 \leq s_2 \leq -1 \wedge -10 \leq s_3 \leq -1 \wedge \mathrm{CP} \equiv \mathrm{CP}^* . \tag{26}$$

QE with REDUCE yields `true`, i.e., the associated eigenvalue placement problem is solvable. If we replace the first inequality by $-5 \leq s_1 \leq -1$ or the last inequality by $-5 \leq s_3 \leq -1$ we would obtain `false`. Using k_1 as a free variable (i.e., we omit the existence quantifier for k_1 in (26)), REDUCE yields 7.1 KiB source code for the equivalent quantifier-free expression. Applying SLFQ to this expression we get the simplified formula

$$\tfrac{94}{4} \leq k_1 \leq \tfrac{98}{4}.$$

We select $k_1 = \frac{96}{5} = 19.2$ as the center of this interval. For the gain k_2 we obtain the bound $224.985 \lessapprox k_2 \leq 225$ resulting from a third order polynomial, which is fulfilled by $k_2 = 224.99$.

The different design goals discussed in Sections IV-A–IV-F are listed in Tab. I. The associated regions in the parameter space (k_1, k_2) are shown in Fig. 1.

G. Dynamic Requirements

Stability is generally not sufficient, at least in practical applications. Thus, additional requirements on the dynamical behavior of the closed-loop system must be considered. These dynamics are correlated to the position of the eigenvalues in the complex plane. Hence, all eigenvalues of the closed-loop system should be in a predefined area in the complex plane. Fig. 2 shows such an area, which balances between robustness, damping and measurement noise.

How these requirements can be considered in control design is now illustrated using the system (11). As already mentioned, the desired characteristic polynomial of a three-dimensional system can be decomposed into a quadratic and a linear term as given in (15). For the following calculations, we consider that one real eigenvalue should between $\alpha_1 = -1$ and $\alpha_2 = -5$ and one complex pair should have a real part in the same range and the imaginary part should be smaller than their real part ($\varphi = \frac{\pi}{4}$). The requirements correspond to conditions on the parameters $\{b, c, d\}$. For the real root, it directly results in $-5 < d < -1$. The roots of the quadratic equation $s^2 + bs + c = 0$ are given by $s_{1/2} = -\frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c}$. To achieve the damping boundary given by φ, the absolute value of the imaginary and real part of the resulting roots need to be equal $b^2 = 2c$. To avoid that the root of the quadratic term became real the condition $c > \frac{b^2}{4}$. That the real parts are between $\alpha_1 = -1$ and $\alpha_2 = -5$ can be achieved with $2 < b < 10$.

If at least one controller pair (k_1, k_2) exists, which fulfills these requirements, is computed with the following prenex formula:

$$\exists k_1, k_2, b, c, d : 1 < d < 5 \wedge 2 < b < 10 \\ \wedge \frac{b^2}{4} < c < \frac{b^2}{2} \wedge \mathrm{CP} \equiv \mathrm{CP}^* . \tag{27}$$

In this case, all variables are quantified and it results simply `true`. To determine the resulting controller area in parameter space of k_1 and k_2 we cancel the quantifier of these variables and the resulting formula

$$\exists b, c, d : 1 < d < 5 \wedge 2 < b < 10 \\ \wedge \frac{b^2}{4} < c < \frac{b^2}{2} \wedge \mathrm{CP} \equiv \mathrm{CP}^* \tag{28}$$

is solved with QE-techniques. As a result the conditions on the controller parameters

$$(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4) \vee (\phi_3 \wedge \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8), \tag{29}$$

with

$$\begin{aligned}
\phi_1 &= -k1 \leq -11 \\
\phi_2 &= k_1^2 + 10k_1 - 2k_2 \leq -1 \\
\phi_3 &= -15k_1^2 + k_1k_2 + 237k_1 - 11k_2 < 870 \\
\phi_4 &= -25k_1^4 + 12k_1^3k_2 - k_1^2k_2^2 - 380k_1^3 + 196k_1^2k_2 \\
&\quad -34k_1k_2^2 + 2k_2^3 - 2119k_1^2 + 832k_1k_2 - 77k_2^2 \\
&\quad -5070k_1 + 1014k_2 < 4394 \\
\phi_5 &= 25k_1 - 2k_2 < 30 \\
\phi_6 &= -k_1^2 - 10k_1 + 2k_2 \leq 1 \\
\phi_7 &= k_2 - 15k_1 \leq -37 \\
\phi_8 &= k_1 \leq 15
\end{aligned}$$

are obtained. These conditions are mapped in Fig. 3. The conditions of (29) are directly due to the system description (11). For more complex systems the resulting prenex formula (28) becomes more complex as well. This significantly affects the computational time.

V. Conclusion

We discussed the existence and computation of stabilizing, static output feedbacks of linear time-invariant systems. To this end, an algorithmic procedure based on quantifier elimination methods is presented. Based on examples known from the literature the introduced conditions and specific computations

TABLE I

GAINS AND EIGENVALUES OF THE EXAMPLE SYSTEM DEPENDING ON THE DESIGN CRITERIA

k_1	k_2	eigenvalues	design criteria
2	50	$s_1 \approx -1.8688202$, $s_{2,3} \approx -0.0655899 \pm 5.172093j$	stabilization, see [4], [33] and Sections IV-A and IV-B
2.9	42	$s_1 \approx -2.8978163$, $s_{2,3} \approx -0.0010918 \pm 3.8070554j$	stabilization, see Remark 1 in Section IV-B
4	75	$s_1 \approx -1.9736481$, $s_{2,3} \approx -1.0131759 \pm 6.0806389j$	robust stabilization, $\Re(s_i) < -1$, see Section IV-C
18	210	$s_1 = -5$, $s_2 = -6$, $s_3 = -7$	real stabilization, $s_i \in \mathbb{R}$, see Sections IV-D and IV-E
19.2	224.99	$s_1 \approx -4.8201872$, $s_2 \approx -4.9497851$, $s_3 \approx -9.4300277$	real interval stabilization, see Section IV-F

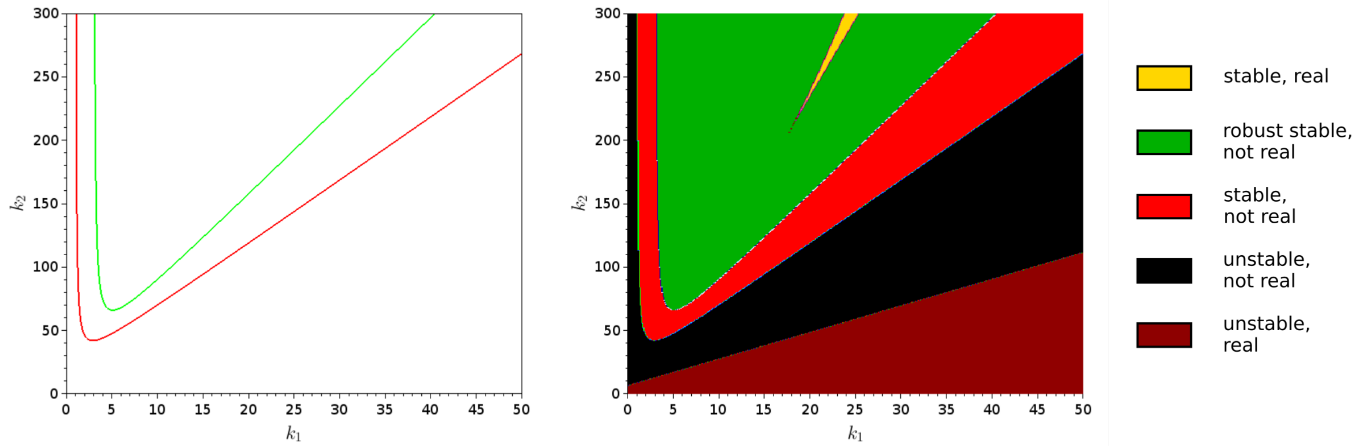

Fig. 1. Bounds resulting from conditions (19) and (22) (left), different regions in the parameter space (k_1, k_2) (right)

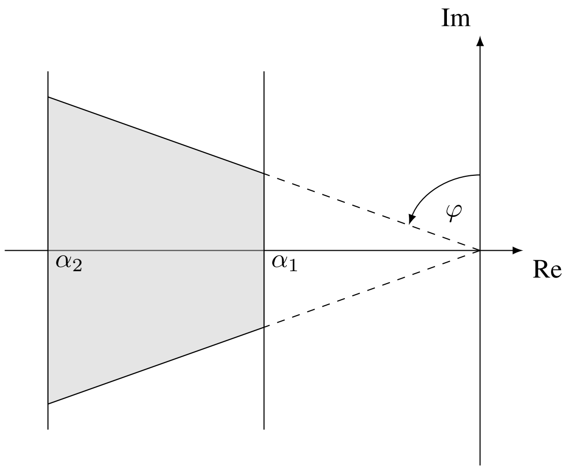

Fig. 2. Destination area of the closed loop eigenvalues in the complex plane

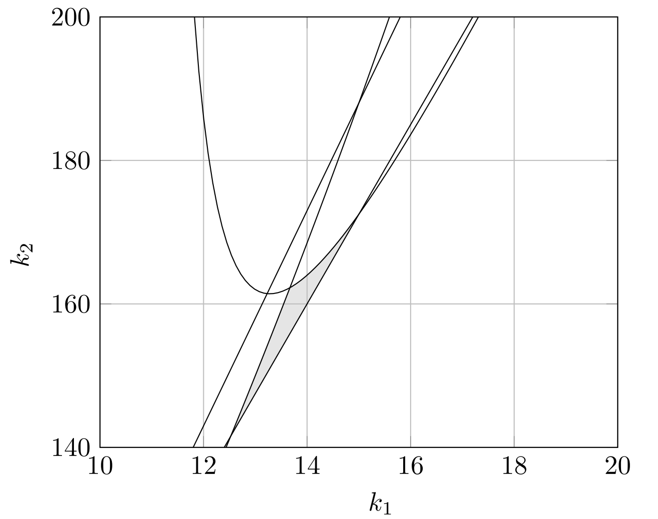

Fig. 3. Resulting controller area fulfilling the given conditions

are illustrated. Furthermore, we considered requirements on the closed-loop eigenvalues, as real stabilizability or damping claims. This gives the main addition to [2].

Contrarily to known techniques, the presented approach gives an exact result due to its real algebraic basis, whereas former approaches use iterative numerical techniques. Additionally, the introduced algorithm provides a straightforward usage and avoids unnecessarily complicated problem formulations. Nevertheless, all QE methods and implementations have inherent computational barriers, which makes a smart problem formulation helpful or even necessary.

ACKNOWLEDGMENT

R. Voßwinkel gratefully acknowledges the financial support of this work by the German Academic Scholarship Foundation.

REFERENCES

[1] R. E. Kalman, "On the general theory of control systems," in *Automatic and remote control*, J. F. Coales, J. R. Ragazzini, and A. T. Fuller, Eds.,

First International Congress of the International Federation of Automatic Control (IFAC), Moscow, 1960. Butterworths and R. Oldenbourg, 1961, pp. 481–492.

[2] K. Röbenack, R. Voßwinkel, and M. Franke, "On the eigenvalue placement by static output feedback via quantifier elimination," in *Mediterranean Conference on Control and Automation (MED'18)*, Zadar, Croatia, Jun. 2018, pp. 133–138.

[3] J. Rosenthal and J. Willems, "Open problems in the area of pole placement," in *Open Problems in Mathematical Systems and Control Theory*, ser. Communication and Control Engineering Series, V. D. Blondel, E. D. Sontag, M. Vidyasagar, and J. C. Willems, Eds. London: Springer–Verlag, 1999, pp. 181–191.

[4] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback — A survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[5] T. Iwasaki and R. E. Skelton, "Parametrization of all stabilizing controllers via quadratic Lyapunov functions," *Journal of Optimization Theory and Applications*, vol. 85, no. 2, pp. 291–307, May 1995.

[6] T. Iwasaki, R. Skelton, and J. Geromel, "Linear quadratic suboptimal control with static output feedback," *Systems & Control Letters*, vol. 23, no. 6, pp. 421–430, 1994.

[7] J. C. Geromel, C. C. de Souza, and R. E. Skelton, "LMI numerical solution for output feedback stabilization," in *American Control Conference*, vol. 1, June 1994, pp. 40–44 vol.1.

[8] K. M. Grigoriadis and R. E. Skelton, "Low-order control design for LMI problems using alternating projection methods," *Automatica*, vol. 32, no. 8, pp. 1117 – 1125, 1996.

[9] P. Finsler, "Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen," *Commentarii mathematici Helvetici*, vol. 9, pp. 188–192, 1937.

[10] F. Zhang, *The Schur complement and its applications*. Springer Science & Business Media, 2006.

[11] P. L. D. Peres, J. C. Geromel, and S. R. Souza, "H_∞ robust control by static output feedback," in *American Control Conference*, June 1993, pp. 620–621.

[12] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

[13] M. Franke, "Eigenvalue assignment by static output feedback – on a new solvability condition and the computation of low gain feedback matrices," *Int. J. Control*, vol. 87, no. 1, pp. 64–75, 2014.

[14] K. Yang and R. Orsi, "Static output feedback pole placement via a trust region approach," *IEEE Transactions on Automatic Control*, vol. 52, no. 11, pp. 2146–2150, Nov 2007.

[15] M. Franke and K. Röbenack, "Calculation of constant output feedback matrices for pole placement by a Gauss-Newton method," *Control and Intelligent Systems*, vol. 42, no. 3, pp. 225–230, 2014.

[16] A. G. Yannakoudakis, "The static output feedback from the invariant point of view," *IMA Journal of Mathematical Control and Information*, vol. 33, no. 3, pp. 639–668, 2016.

[17] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, 2nd ed. Berlin, Heidelberg: Springer, 2006.

[18] A. Tarski, *A Decision Method for a Elementary Algebra and Geometry*, ser. Project rand. Rand Corporation, 1948.

[19] G. E. Collins, "Quantifier elimination for real closed fields by cylindrical algebraic decomposition–preliminary report," *ACM SIGSAM Bulletin*, vol. 8, no. 3, pp. 80–90, 1974.

[20] V. Weispfenning, "The complexity of linear problems in fields," *Journal of Symbolic Computation*, vol. 5, no. 1-2, pp. 3–27, 1988.

[21] R. Loos and V. Weispfenning, "Applying linear quantifier elimination," *The Computer Journal*, vol. 36, no. 5, pp. 450–462, 1993.

[22] V. Weispfenning, "Quantifier elimination for real algebra – the cubic case," in *Proc. of the Int. Symp. on Symbolic and Algebraic Computation (ISSAC)*. ACM, 1994, pp. 258–263.

[23] L. Gonzalez-Vega, H. Lombardi, T. Recio, and M.-F. Roy, "Sturm-Habicht sequence," in *Proc. of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation*. ACM, 1989, pp. 136–146.

[24] L. Yang, X. R. Hou, and Z. B. Zeng, "Complete discrimination system for polynomials," *Science in China Series E Technological Sciences*, vol. 39, no. 6, 1996.

[25] H. Iwane, H. Yanami, H. Anai, and K. Yokoyama, "An effective implementation of symbolicnumeric cylindrical algebraic decomposition for quantifier elimination," *Theoretical Computer Science*, vol. 479, pp. 43–69, 2013.

[26] G. E. Collins and H. Hong, "Partial cylindrical algebraic decomposition for quantifier elimination," *Journal of Symbolic Computation*, vol. 12, no. 3, pp. 299–328, 1991.

[27] C. W. Brown, "QEPCAD B: A program for computing with semi-algebraic sets using CADs," *ACM SIGSAM Bulletin*, vol. 37, no. 4, pp. 97–108, 2003.

[28] A. Dolzmann and T. Sturm, "Redlog: Computer algebra meets computer logic," *ACM SIGSAM Bulletin*, vol. 31, no. 2, pp. 2–9, 1997.

[29] M. Košta, "New concepts for real quantifier elimination by virtual substitution," Dissertation, Universität des Saarlandes, Fakultät für Mathematik und Informatik, Saarbrücken, Germany, 2016.

[30] C. W. Brown and C. Gross, "Efficient preprocessing methods for quantifier elimination," in *CASC*, ser. Lecture Notes in Computer Science, vol. 4194. Springer, 2006, pp. 89–100.

[31] C. Chen and M. M. Maza, "Quantifier elimination by cylindrical algebraic decomposition based on regular chains," *Journal of Symbolic Computation*, vol. 75, pp. 74–93, 2016.

[32] H. Yanami and H. Anai, "The Maple package SyNRAC and its application to robust control design," *Future Generation Computer Systems*, vol. 23, no. 5, pp. 721–726, 2007.

[33] B. Anderson, N. Bose, and E. Jury, "Output feedback stabilization and related problems-solution via decision methods," *IEEE Trans. on Automatic Control*, vol. 20, no. 1, pp. 53–66, Feb. 1975.

[34] C. T. Abdallah, P. Dorato, R. Liska, S. Steinberg, and W. Yang, "Applications of quantifier elimination theory to control theory," University of New Mexico, Tech. Rep., Sep. 1995.

[35] P. Dorato, "Quantified multivariable polynomial inequalities –The mathematics of practical control design problems," *IEEE Control Systems Magazine*, vol. 20, no. 5, pp. 48–58, Oct. 2000.

[36] J. Tong and N. Bajcinca, "Computation of feasible parametric regions for Lyapunov functions," in *Asian Control Conference (ASCC)*, 2017.

[37] R. Voßwinkel, J. Tong, K. Röbenack, and N. Bajcinca, "Stability bounds for systems and mechanisms in linear descriptor form," in 59^{th} *Ilmenau Scientific Colloquium*, Sep. 2017.

[38] H. Anai and S. Hara, "A parameter space approach to fixed-order robust controller synthesis by quantifier elimination," *Int. J. Control*, vol. 79, no. 11, pp. 1321–1330, 2006.

[39] N. Hyodo, M. Hong, H. Yanami, S. Hara, and H. Anai, "Solving and visualizing nonlinear parametric constraints in control based on quantifier elimination," *Applicable Algebra in Engineering, Communication and Computing*, vol. 18, no. 6, pp. 497–512, Dec 2007. [Online]. Available: https://doi.org/10.1007/s00200-007-0056-7

[40] Z. She, B. Xia, R. Xiao, and Z. Zheng, "A semi-algebraic approach for asymptotic stability analysis," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 588–596, 2009.

[41] R. Voßwinkel, K. Röbenack, and N. Bajcinca, "Input-to-state stability mapping for nonlinear control systems using quantifier elimination," in *European Control Conference (ECC)*, Limassol, Cyprus, Jun. 2018, pp. 906–911.

[42] K. Röbenack, R. Voßwinkel, and H. Richter, "Automatic generation of bounds for polynomial systems with application to the Lorenz system," *Chaos, Solitons & Fractals*, vol. 113C, pp. 25–30, 2018.